Journal of Engineering Mathematics, Vol. 4, No. 4, October 1970
Wolters-Noordhoff Publishing–Groningen
Printed in the Netherlands

On Graphs and Rigidity of Plane Skeletal Structures

G. LAMAN

Department of Mathematics, University of Amsterdam, The Netherlands

(Received June 8, 1970)

SUMMARY

In this paper the combinatorial properties of rigid plane skeletal structures are investigated. Those properties are found to be adequately described by a class of graphs.

1. Introduction

The subject of rigidity of skeletal structures is treated in more or less detail in handbooks on mechanics and on statics. The treatment one finds there clearly may be useful to engineers who have to solve statical problems concerning given structures either with a very limited number of components or having them conveniently arranged according to some regular pattern. If however one is confronted with structures not meeting those requirements the approach of the handbooks is no longer adequate at all.

Moreover the set of notions found in this literature does not lend itself easily to a mathematical treatment. If definitions are given at all they tend to excel by ambiguity and obscurity. For the purpose of this paper clear definitions are indispensable. That is why we give new definitions of skeletal structure and of rigidity (in Section 2). The word skeletal structure is chosen so as to avoid some misunderstandings connected with the other more or less current designations truss, framework, articulated structure.

In Section 3 examples are given which it is hoped will elucidate the definitions and give the reader the opportunity to materialize the rather abstract goings-on in later paragraphs.

In Section 4 some preliminary analytic geometry is set forth. Section 5 treats criteria for rigidity and a class of graphs is found which are closely related to rigid skeletal structures. This class of graphs is shown to be obtainable by an algorithm and its relation to rigid skeletal structures is examined more thoroughly in Section 6.

2. Graphs, Skeletal Structures and Rigidity

In the following we shall have to use part of the language of set-theory if only for the sake of terseness. As many readers are not supposed to be familiar with this language we collect here the items we need.

$s \in S$	means:	the element s belongs to the set S;
$s \notin S$	means:	the element s does not belong to the set S.
$S \subset T$	means:	every element of S belongs to T or: S is a subset of T.
$S = \{a, b, c, d\}$	means:	S contains the elements a, b, c, d and no other elements.
$S = \{a_j \mid i = 1, 2, \ldots, n\}$	means:	S consists of the elements $a_1, a_2, \ldots, a_n$. If $I = \{1, 2, \ldots, n\}$ this may be written: $S = \{a_i \mid i \in I\}$.

If S and T are both sets we denote by $S \bigcup T$ the set of elements belonging to either S or T or both, by $S \bigcap T$ the set of elements belonging to both S and T, by $S \setminus T$ the set of those elements of S which do not belong to T.

If $t \in S$ we shall also write $S \setminus t$ for the set which remains when t is taken out of S.

$|S|$ will denote the number of elements of a finite set S and $\mathscr{P}(S)$ the set of pairs of S, i.e. the set having as it elements those subsets of S which contain exactly two different elements of S. So we have $|\mathscr{P}(S)| = \frac{1}{2}|S|(|S|-1)$.

Remark: We shall also use $|x|$ for the length of a Euclidean vector x. There will arise no confusion between $|x|$ and $|S|$.

Our definition of the concept of graph is adapted to the needs of this paper (in the language of more general definitions our graphs are simple, undirected, and finite).

Definition 2.1: *A graph Γ consists of a finite set K (the set of* nodes *of Γ) and a subset $\mathscr{R}$* of *$\mathscr{P}(K)$ (the set of* edges *of Γ). Notation: $\Gamma(K, \mathscr{R})$.*

The degree *of a node $a \in K$ is the number of those $b \in K$ for which $(a, b) \in \mathscr{R}$ (the* neighbours *of a).*

Definition 2.2: *A plane* skeletal structure *consists of a graph Γ and a mapping χ of K into the Euclidean plane E^2 satisfying $\chi a \neq \chi b$ if $(a, b) \in \mathscr{R}$. Notation: (Γ, χ).*

If a is a node of Γ then χa is called a joint *of the skeletal structure (Γ, χ) and if (a, b) is an edge of Γ then the straight line segment between χa and χb is called a* bar *of (Γ, χ). Γ is called the* underlying *graph of (Γ, χ) and (Γ, χ) is a* plane Euclidean realization *of Γ (cf. remark following example 2.1).*

Comment (in the sequel this will mean each time a helping hand to those who may feel our approach to be too abstract): Definition 2.2 might be formulated:

A plane skeletal structure is a graph (Γ) together with the assignment (χ) of a point (χa) of the plane to every node (a) of the graph in such a way as to avoid the same point to be assigned to two neighbouring nodes (i.e. to avoid bars having length 0).

Definition 2.3: *A* length-preserving displacement *of a plane skeletal structure (Γ, χ) consists of:*

a) *A segment $[\beta, \gamma]$ of real numbers with $0 \in [\beta, \gamma]$;*

b) *for every $a \in K$ and for every $\tau \in [\beta, \gamma]$ a point $\chi_\tau a \in E^2$, satisfying the following conditions:*
   1) *$\chi_0 a = \chi a$ for every a;*
   2) *the function $\chi_\tau a$ is differentiable for every a;*
   3) *(Γ, χ_τ) is a plane skeletal structure for every τ;*
   4) *$|\chi_\tau a - \chi_\tau b| = |\chi a - \chi b|$ for every τ and every $(a, b) \in \mathscr{R}$.*

Comment: A length-preserving displacement is a set of plane skeletal structures (depending differentiably on time τ) which leaves lengths of bars invariant (4) and of which the given skeletal structure is the one for $\tau = 0$ (1).

A motion of E^2 in itself is a function π of time τ and of $x \in E^2$, differentiable in τ and taking its values $\pi_\tau x$ in E^2 in such a way that for every x: $\pi_0 x = x$ and for every τ and for every x_1 and x_2: $|\pi_\tau x_1 - \pi_\tau x_2| = |x_1 - x_2|$.

A length-preserving displacement is said to be trivial if it results from some motion of E^2 in itself, i.e. if there is a π satisfying $\pi_\tau \chi = \chi_\tau$.

Definition 2.4: *An* infinitesimal displacement *of a plane skeletal structure is a map μ of K into the 2-dimensional Euclidean vectorspace $\mathbb{R}^2$.*

Comment: An infinitesimal displacement of a plane skeletal structure is the assignment (μ) of a vector (μa) to every joint (χa). One might think of this vector as velocity.

The set $M(K)$ of all infinitesimal displacements of a given skeletal structure is made into a $2|K|$-dimensional vectorspace by the following definitions:

a) For every real α and every $\mu \in M(K)$ $\alpha\mu$ is defined to be the element of $M(K)$ satisfying:

$$(\alpha\mu)a = \alpha \cdot \mu a \quad \text{for every} \quad a \in K\ ;$$

b) For every pair $\mu_1 \in M(K)$ and $\mu_2 \in M(K)$ $\mu_1 + \mu_2$ is defined to be the element of $M(K)$ satisfying:

$$(\mu_1 + \mu_2)a = \mu_1 a + \mu_2 a \quad \text{for every} \quad a \in K\ .$$

Definition 2.5: *A* small displacement *of a plane skeletal structure consists of*:
a) *An infinitesimal displacement* μ;
b) *a real number* $\alpha > 0$;
c) *for every real* τ *with* $|\tau| \leqq \alpha$ *a map* $\chi_\tau: K \to E^2$ *satisfying* $\chi_\tau a = \chi a + \tau \cdot \mu a + o(\tau)$ *for every a K*. (*Here* $o(\tau)$ *has the usual meaning*: $\lim_{\tau\to 0} \tau^{-1} \cdot o(\tau) = 0$)

Comment: Definition 2.5 is a local version of definition 2.3 in a neighbourhood of $\tau = 0$ without the condition of length-preservation. This condition reappears in local form in the following definition where admissibility means that under the small displacement lengths of bars do not change up till terms of higher than first order in τ.

Definition 2.6: *A small displacement is* admissible *if for every* $(a, b) \in \mathscr{R}$

$$|\chi_\tau a - \chi_\tau b| - |\chi a - \chi b| = o(\tau).$$

As is well known an infinitesimal motion of E^2 in itself is a map $\psi: E^2 \to \mathbb{R}^2$ satisfying $(\psi x_1 - \psi x_2, x_1 - x_2) = 0$ for every pair x_1, x_2 of E^2. ψ may be identified with the velocity field of some motion π of E^2 at time $\tau = 0$. The vanishing of the inner product means that the difference of the velocities ψx_1 and ψx_2 is perpendicular to the line segment between x_1 and x_2.

Definition 2.7: *An infinitesimal displacement* μ *of* (Γ, χ) *is* trivial *if there exists an infinitesimal motion* ψ *satisfying* $\psi\chi = \mu$.

Definition 2.8: *A plane skeletal structure is* rigid *if every admissible small displacement has a trivial infinitesimal displacement.*

Comment: Together those definitions amount to:
A plane skeletal structure is rigid if for every admissible small displacement its infinitesimal displacement assigns the same vectors to the joints of the structure as does some infinitesimal motion.

3. Examples

Example 3.1 (fig. 1a, 1b): $K = \{a_1, a_2, a_3\}$, $\mathscr{R} = \{(a_1, a_2), (a_2, a_3), (a_3, a_1)\}$, $\Gamma = \Gamma(K, \mathscr{R})$.
$\chi^1 a_1 = (1, 0)$, $\chi^1 a_2 = (0, 1)$, $\chi^1 a_3 = (-1, 0)$; (Γ, χ^1) is rigid.
$\chi^2 a_1 = (1, 0)$, $\chi^2 a_2 = (0, 0)$, $\chi^2 a_3 = (-1, 0)$; (Γ, χ^2) is not rigid.

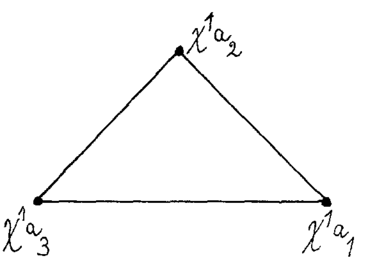

Figure 1a.

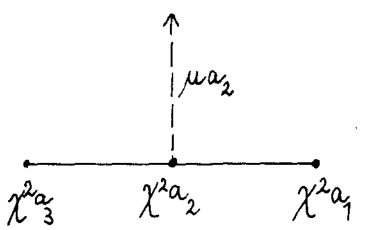

Figure 1b.

Indeed choose $\mu a_1 = (0, 0)$, $\mu a_2 = (0, 1)$, $\mu a_3 = (0, 0)$, then χ_τ^2 defined by $\chi_\tau^2 a_i = \chi^2 a_i + \tau \cdot \mu a_i$ for $i = 1, 2, 3$ is an admissible small displacement with non-trivial infinitesimal displacement as is easily verified.

This simple example already shows that a rigid and a non-rigid skeletal structure may have the same underlying graph.

Remark: If for any (Γ, χ) all joints are collinear there apparently exist admissible small displacements with non-trivial infinitesimal displacements. To avoid this type of exception we shall assume henceforth that χ is non-degenerate i.e. that not all joints are contained in a straight line in E^2.

Example 3.2: $K=\{a_1, a_2, a_3, a_4\}$, $\mathscr{R}=\{(a_1, a_2), (a_2, a_3), (a_3, a_4), (a_4, a_1)\}$, $\Gamma=\Gamma(K, \mathscr{R})$.

Whatever plane realization one chooses it is always possible to find an admissible small displacement with non-trivial infinitesimal displacement. This example shows the existence of graphs not admitting a rigid realization.

Example 3.3 (fig. 2a, 2b): $K=\{a_i | i=1, 2, \ldots, 6\}$, $\mathscr{R}=\{(a_1, a_2), (a_2, a_3), (a_3, a_1), (a_1, a_4), (a_2, a_5), (a_3, a_6), (a_4, a_5), (a_5, a_6), (a_6, a_4)\}$, $\Gamma=\Gamma(K, \mathscr{R})$.

$\chi^1 a_1=(1, 0)$, $\chi^1 a_2=(0, 1)$, $\chi^1 a_3=(-1, 0)$, $\chi^1 a_4=(1, 2)$, $\chi^1 a_5=(0, 3)$, $\chi^1 a_6=(-1, 2)$.
The skeletal structure (Γ, χ^1) admits a length-preserving displacement and *a fortiori* is non-rigid; indeed we may take $\chi^1_\tau a_i=\chi^1 a_i$ for $i=1, 2, 3$ and for $i=4, 5, 6$: $\chi^1_\tau a_i=\chi^1 a_i+2(\sin\tau, 1-\cos\tau)$.

A small change however produces a rigid (Γ, χ^2), e.g. $\chi^2 a_i=\chi^1 a_i$ for $i=1, 2, 3, 4, 6$ and $\chi^2 a_5=(\delta, 3)$ with $\delta\neq 0$.

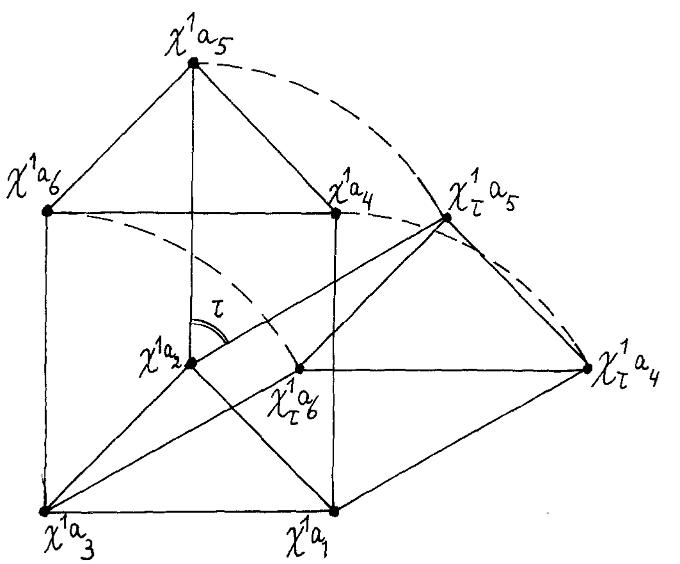

Figure 2a.

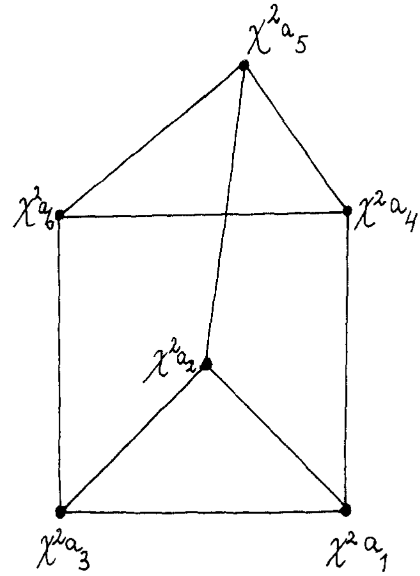

Figure 2b.

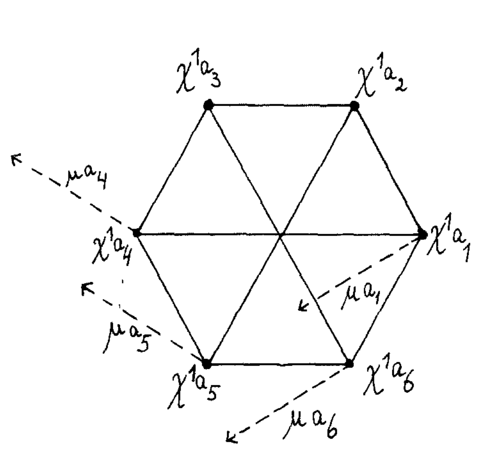

Figure 3a.

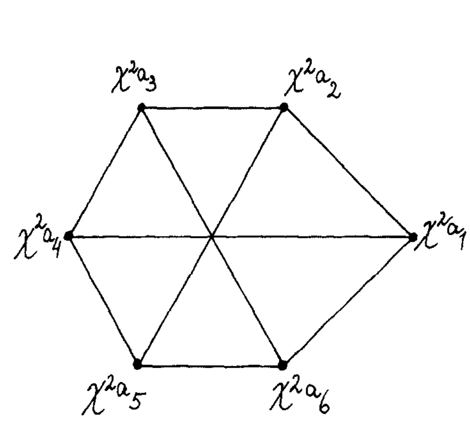

Figure 3b.

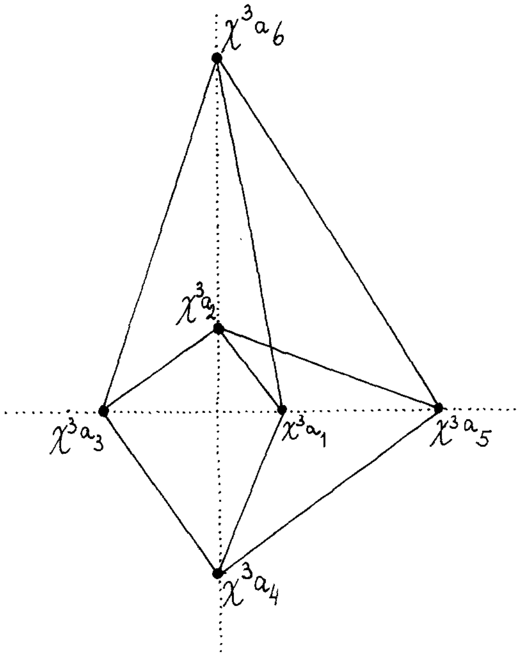

Figure 3c.

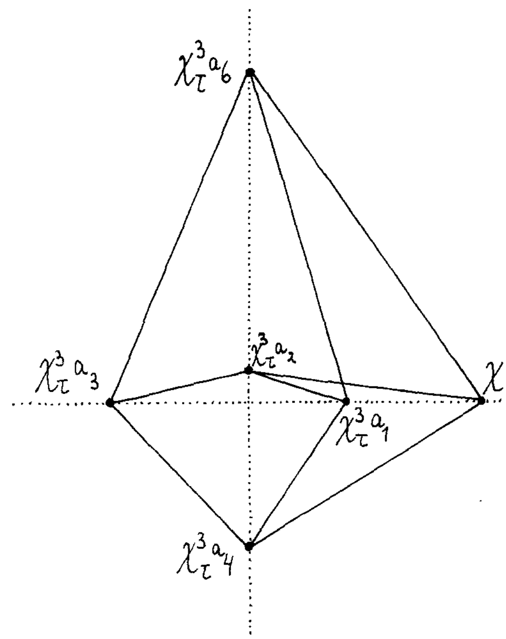

Figure 3d.

Example 3.4 (fig. 3a, 3b): $\Gamma = \Gamma(K, \mathscr{R})$ with $K = \{a_i \mid i = 1, 2, \ldots, 6\}$ and $\mathscr{R} = \{(a_1, a_2), (a_2, a_3), (a_3, a_4), (a_4, a_5), (a_5, a_6), (a_6, a_1), (a_1, a_4), (a_2, a_5), (a_3, a_6)\}$. (Γ, χ^1) is not rigid if χ^1 is defined by $\chi^1 a_1 = (1, 0)$, $\chi^1 a_2 = (\frac{1}{2}, \frac{1}{2}\sqrt{3})$, $\chi^1 a_3 = (-\frac{1}{2}, \frac{1}{2}\sqrt{3})$, $\chi^1 a_4 = (-1, 0)$, $\chi^1 a_5 = (-\frac{1}{2}, -\frac{1}{2}\sqrt{3})$, $\chi^1 a_6 = (\frac{1}{2}, -\frac{1}{2}\sqrt{3})$.

Indeed define μ by $\mu a_2 = \mu a_3 = 0$, $\mu a_1 = \mu a_6 = (-\frac{1}{2}\sqrt{3}, -\frac{1}{2})$, $\mu a_4 = \mu a_5 = (-\frac{1}{2}\sqrt{3}, \frac{1}{2})$ and χ^1_τ by $\chi^1_\tau a_i = \chi^1 a_i + \tau \cdot \mu a_i$ then χ^1_τ is admissible and μ non-trivial. If $\chi^2 a_i = \chi^1 a_i$ for $i = 2, 3, 4, 5, 6$ and $\chi^2 a_1 = (\delta, 0)$, $\delta \neq 1$, then (Γ, χ^2) is rigid.

As in example 3.3 there exists a realization of Γ admitting a length-preserving displacement (Cf. fig. 3c, 3d; I owe this to Prof. F. A. Muller).

To this end χ^3 is defined as follows: $\chi^3 a_1 = (\xi_1, 0)$, $\chi^3 a_3 = (\xi_3, 0)$, $\chi^3 a_5 = (\xi_5, 0)$, $\chi^3 a_2 = (0, \eta_2)$, $\chi^3 a_4 = (0, \eta_4)$, $\chi^3 a_6 = (0, \eta_6)$ where $\xi_1, \xi_3, \xi_5, \eta_2, \eta_4, \eta_6$ are non-zero real numbers.

A length-preserving displacement of (Γ, χ^3) is now given by

$$\chi^3_\tau a_i = (\xi_i(1+\xi_i^{-2}\tau)^{\frac{1}{2}}, 0), \quad i = 1, 3, 5 \text{ and } \chi^3 a_i = (0, \eta_i(1-\eta_i^{-2}\tau)^{\frac{1}{2}}), \quad i = 2, 4, 6 .$$

4. Trivial and Non-Trivial Infinitesimal Displacements

Let $x_i = (\xi_i, \eta_i)$ be points of E^2 and u_i vectors of $\mathbb{R}^2 (i = 0, 1, 2, \ldots, n)$.

We shall denote the matrix

$$\begin{pmatrix} \xi_0 - \xi_1 & \eta_0 - \eta_1 & (u_1, x_1 - x_0) \\ \xi_0 - \xi_2 & \eta_0 - \eta_2 & (u_2, x_2 - x_0) \\ \vdots & \vdots & \vdots \\ \xi_0 - \xi_n & \eta_0 - \eta_n & (u_n, x_n - x_0) \end{pmatrix}$$

by $(x_0 - x_i \quad (u_i, x_i - x_0))_{i=1,2,\ldots,n}$ and the determinant

$$\begin{vmatrix} \xi_0 - \xi_k & \eta_0 - \eta_k & (u_k, x_k - x_0) \\ \xi_0 - \xi_l & \eta_0 - \eta_l & (u_l, x_l - x_0) \\ \xi_0 - \xi_m & \eta_0 - \eta_m & (u_m, x_m - x_0) \end{vmatrix}$$

by

$$| x_0 - x_i \quad (u_i, x_i - x_0) |_{i=k,l,m} .$$

Proposition 4.1: *If the inhomogeneous quadratic form* $a_{11}\xi^2+a_{12}\xi\eta+a_{22}\eta^2+b_1\xi+b_2\eta+c$ *vanishes at 3 non-collinear points* x_i *of* E^2 *and also at the midpoints of every pair of them, then the quadratic form vanishes identically.*

Proof: Homogenize the form by introduction of a new variable ζ and then transform to barycentric coordinates with respect to the 3 given points x_i. The coefficients of the form in those coordinates are easily seen to be zero by substitution first of (1, 0, 0), (0, 1, 0) and (0, 0, 1) and after that of (0, 1, 1), (1, 0, 1) and (1, 1, 0).

Proposition 4.2: *If* x_i $(i=1, 2, 3)$ *are non-collinear points of* E^2 *and* u_i 3 *vectors of* $\mathbb{R}^2$ *then the following assertions are equivalent*:
A. $(x_i-x_j, u_i-u_j)=0$ *for every i and j*;
B. $|x-x_i \quad (u_i, x_i-x)|_{i=1,2,3}=0$ *identically in x.*

Proof: First suppose *B*. Substitution of $2x=x_i+x_j$ $(i\neq j)$ into $|x-x_k \quad (u_k, x_k-x)|_{k=1,2,3}=0$ and addition of the *i*-th row to the *j*-th row yields $(u_i-u_j, x_i-x_j)\cdot D=0$ where D is a determinant which vanishes if and only if x_1, x_2 and x_3 are collinear. So we have proved *A*.

Now suppose *A*. Consider $|x-x_i \quad (u_i, x_i-x)|_{i=1,2,3}$ as a polynomial in the coordinates ξ, η of x.

As $\begin{vmatrix} \xi-\xi_1 & \eta-\eta_1 \\ \xi-\xi_2 & \eta-\eta_2 \end{vmatrix}=0$

is the equation of the line though x_1 and x_2, the first member is of degree 1 etc., so the polynomial is a quadratic form. This inhomogeneous quadratic form vanishes at x_i and at $2^{-1}(x_i+x_j)$ and so satisfies the conditions of proposition 4.1.

Hence *A* implies *B*.

Lemma 4.3: *If* x_i $(i=1, 2, 3)$ *are non-collinear points of* E^2 *and* u_1, u_2 *vectors of* $\mathbb{R}^2$ *then the system of equations*

$$(x_1-x_3, u_1-u)=0$$
$$(x_2-x_3, u_2-u)=0$$

has exactly one solution.

Proof: The determinant of the system is non-zero.

Proposition 4.4: *An infinitesimal displacement* μ *of* (Γ, χ) *is trivial if and only if the matrix*

$$(x-\chi a \quad (\mu a, \chi a-x))_{a\in K}$$

has rank 2 *identically in x. The trivial infinitesimal displacements constitute a* 3*-dimensional subspace of the* $2|K|$*-dimensional vectorspace of infinitesimal displacements.*

Proof: First suppose the rank of $(x-\chi a \quad (\mu a, \chi a-x))_{a\in K}$ to be 2 identically in x. Choose 3 non-collinear joints of χK: χk_1, χk_2 and χk_3 and put $\psi\chi k_1=\mu k_1$ and $\psi\chi k_2=\mu k_2$—the infinitesimal motion ψ satisfying those conditions is clearly unique—.

Then by proposition 3.2 it follows easily that both μk_3 and $\psi\chi k_3$ satisfy

$$(\chi k_1-\chi k_3, \mu k_1-u)=0$$
$$(\chi k_2-\chi k_3, \mu k_2-u)=0\,.$$

Now lemma 3.3 yields $\psi\chi k_3=\mu k_3$.

For every other $k\in K$ there exist two joints among χk_1, χk_2, χk_3 such that χk is not collinear with them. Repetition of the above argument then yields $\psi\chi k=\mu k$ for all $k\in K$. So μ is trivial.

If on the other hand μ is trivial then it follows easily from the existence of an infinitesimal motion ψ and from proposition 3.2 that the matrix $(x-\chi a \quad (\mu a, \chi a - x))_{a\in K}$ has rank 2 for every x.

Finally it is clear that there is a one-to-one correspondence between the infinitesimal motions and the trivial infinitesimal displacements and so the trivial infinitesimal displacements indeed constitute a 3-dimensional subspace of the $2|K|$-dimensional vectorspace of infinitesimal displacements.

5. Criteria for Rigidity

Proposition 5.1: *A small displacement of a 2-dimensional skeletal structure is admissible if and only if the corresponding infinitesimal displacement satisfies $(\chi a - \chi b, \mu a - \mu b) = 0$ for every $(a, b) \in \mathscr{R}$.*

Proof:

$$
\begin{aligned}
&|\chi_\tau a - \chi_\tau b| - |\chi a - \chi b| = \\
&= |\chi a - \chi b + \tau(\mu a - \mu b) + o(\tau)| - |\chi a - \chi b| = \\
&= (\chi a - \chi b + \tau(\mu a - \mu b) + o(\tau), \chi a - \chi b + \tau(\mu a - \mu b) + o(\tau))^{\frac{1}{2}} - (\chi a - \chi b, \chi a - \chi b)^{\frac{1}{2}} = \\
&= \{(\chi a - \chi b, \chi a - \chi b) + 2\tau(\chi a - \chi b, \mu a - \mu b) + o(\tau)\}^{\frac{1}{2}} - (\chi a - \chi b, \chi a - \chi b)^{\frac{1}{2}} = \\
&= |\chi a - \chi b| \left[\left\{ 1 + 2\tau \frac{(\chi a - \chi b, \mu a - \mu b)}{|\chi a - \chi b|^2} + o(\tau) \right\}^{\frac{1}{2}} - 1 \right] = \\
&= |\chi a - \chi b| \left[1 + \tfrac{1}{2} \cdot 2\tau \frac{(\chi a - \chi b, \mu a - \mu b)}{|\chi a - \chi b|^2} + o(\tau) - 1 \right] = \\
&= \tau \frac{(\chi a - \chi b, \mu a - \mu b)}{|\chi a - \chi b|} + o(\tau).
\end{aligned}
$$

From this and definition 1.6 the proposition follows.

A μ satisfying $(\chi a - \chi b, \mu a - \mu b) = 0$ for all $(a, b) \in \mathscr{R}$ will be called an *admissible infinitesimal displacement.*

Theorem 5.2: *The assertions A, B, C, D are equivalent:*

A. The 2-dimensional skeletal structure (Γ, χ) is rigid.

B. If $(\chi a - \chi b, \mu a - \mu b) = 0$ for all $(a, b) \in \mathscr{R}$, then μ is trivial.

C. If $(\chi a - \chi b, \mu a - \mu b) = 0$ for all $(a, b) \in \mathscr{R}$, then $(\chi a - \chi b, \mu a - \mu b) = 0$ for all $(a, b) \in \mathscr{P}(K)$.

D. The matrix of the system of equations:

$$(\chi a - \chi b, \mu a - \mu b) = 0 \quad \text{for all} \quad (a, b) \in \mathscr{R}$$

has rank $2|K| - 3$.

Proof: The equivalence of A and B follows from the definition of rigidity and from proposition 5.1. The equivalence of B and C follows from proposition 4.4. The equivalence of B and D follows from the second part of proposition 4.4.

From the above it is clear that for a rigid skeletal structure a necessary condition is: $|\mathscr{R}| \geqq 2|K| - 3$. This explains the non-rigidity of every realization of example 3.2.

Proposition 5.3: *Suppose $\Gamma(K, \mathscr{R})$ to be a graph, a, b_1, b_2 to belong to K, $(a, b_1) \in \mathscr{R}$, $(a, b_2) \in \mathscr{R}$ and $(a, k) \notin \mathscr{R}$ if $k \neq b_i$. Let $K' = K \backslash a$, $\mathscr{R}' = \mathscr{R} \cap \mathscr{P}(K') = \mathscr{R} \backslash \{(a, b_1), (a, b_2)\}$ and $\Gamma' = \Gamma'(K', \mathscr{R}')$.*

Let moreover (Γ', χ') be a rigid plane realization of Γ'. Then (Γ, χ) is a rigid plane realization of Γ where χ satisfies $\chi k = \chi' k$ for every $k \in K'$ and χa is non-collinear with χb_1 and χb_2.

Proof: This is an easy consequence of lemma 4.3.

Proposition 5.4: *Suppose $\Gamma(K, \mathscr{R})$ to be a graph, a, b_1, b_2, b_3 to belong to K, $(a, b_i)\in\mathscr{R}$ for $i=1, 2, 3$, $(b_1, b_2)\notin\mathscr{R}$ and $(a, k)\notin\mathscr{R}$ if $k\neq b_i$. Let $K'=K\backslash a$, $\mathscr{R}'=\mathscr{R}\bigcap\mathscr{P}(K')=\mathscr{R}\backslash\{(a, b_1), (a, b_2), (a, b_3)\}$, $\Gamma'=\Gamma'(K', \mathscr{R}')$ and $\Gamma''=\Gamma''(K', \mathscr{R}'\bigcup\{(b_1, b_2)\})$. Let moreover (Γ'', χ') be rigid. Then there is a χ: $K\to E^2$ satisfying $\chi k=\chi' k$ for every $k\in K'$ and such that (Γ, χ) is rigid.*

Proof: Assume (Γ', χ') to be non-rigid; the other case is easy. The admissible infinitesimal displacements of (Γ'', χ') are $\alpha_1\lambda^1+\alpha_2\lambda^2+\alpha_3\lambda^3$ with $\lambda^1, \lambda^2, \lambda^3$ trivial and $\alpha_i\in\mathrm{R}$ and therefore those of (Γ', χ') are $\beta_1\lambda^1+\beta_2\lambda^2+\beta_3\lambda^3+\beta_4\lambda^4$ with λ^4 non-trivial so $|x-\chi b_j \quad (\lambda^4 b_j, \chi b_j-x)|_{j=1,2,3}=0$ does not hold identically in x.

For any choice of χa any admissible displacement μ of (Γ, χ) should satisfy $\mu=\Sigma_{i=1}^4\beta_i\mu^i$ with $\mu^i k=\lambda^i k$ for $k\in K'$ and $(\chi a-\chi b_j, \mu a-\mu b_j)=0$ for $j=1, 2, 3$. The necessary and sufficient condition for solvability of this system is (proposition 4.2)

$$|\chi a-\chi b_j \quad (\mu b_j, \chi b_j-\chi a)|_{j=1,2,3}=0\,.$$

Now $\mu b_j=\Sigma_{i=1}^4\beta_i\lambda^i b_j$ and

$$|\chi a-\chi b_j \quad (\lambda^i b_j, \chi b_j-\chi a)|_{j=1,2,3}=0$$

identically in χa for $i=1, 2, 3$ because $\lambda^1, \lambda^2, \lambda^3$ are trivial, so the condition may be replaced by

$$\beta_4|\chi a-\chi b_j \quad (\lambda^4 b_j, \chi b_j-\chi a)|_{j=1,2,3}=0\,.$$

Choose χa off the quadratic curve

$$|x-\chi b_j \quad (\lambda^4 b_j, \chi b_j-x)|_{j=1,2,3}=0$$

so as to make

$$|\chi a-\psi b_j \quad (\lambda^4 b_j, \chi b_j-\chi a)|_{j=1,2,3}\neq 0\,.$$

Then β_4 is zero and μ is trivial, so χ meets the requirements mentioned in the proposition.

Proposition 5.5: *Every rigid plane skeletal structure (Γ, χ) has a rigid substructure with $|K|$ joints and $2|K|-3$ bars.*

Proof: This follows from theorem 5.2, D; the matrix of $(\chi a-\chi b, \mu a-\mu b)=0$ for all $(a, b)\in\mathscr{R}$ has rank $2|K|-3$. We can drop equations (and corresponding $(a, b)\in\mathscr{R}$) until only $2|K|-3$ independent equations are left.

If K' is a subset of the set of nodes K having at least two elements then there is a graph having K' as set of nodes and as edges exactly those of $\mathscr{R}$ of which both nodes belong to K'. This graph is called the *subgraph spanned by K'*.

Theorem 5.6: *Any rigid plane skeletal structure (Γ, χ) with $|\mathscr{R}|=2|K|-3$ has the* property E: *If $K'\subset K$ and $|K'|\geqq 2$ then $|\mathscr{R}\bigcap\mathscr{P}(K')|\leqq 2|K'|-3$.*

Comment: Property E says that in every spanned subgraph the number of edges is less than or equal to two times the number of nodes minus three.

Proof: To every $(a, b)\in\mathscr{R}$ there corresponds an equation $(\chi a-\chi b, \mu a-\mu b)=0$; because of rigidity and proposition 5.5 those equations are independent. If for some $K'\subset K$ with $|K'|\geqq 2$ we should have $|\mathscr{R}\bigcap\mathscr{P}(K')|>2|K'|-3$ then by proposition 5.5 there would be dependence among the corresponding $2|K'|-3$ equations, which is impossible.

A graph with property E, $|K|\geqq 3$ and $|\mathscr{R}|=2|K|-3$ we shall call an *E-graph*. Now we may summarize the above results: Every rigid plane skeletal structure contains a rigid realization of an E-graph. To a large extent this result is reversible and the principal outcome of this paper is: Every E-graph has rigid realizations (theorem 6.5). The proof of this theorem requires some propositions on the structure of E-graphs, which will be proved in the next section.

6. E-graphs

Proposition 6.1: *An E-graph does not contain nodes of degree* 1 *and at least one node of degree* $\leqq 3$

Proof: Suppose $\Gamma(K, \mathscr{R})$ is an E-graph and a is a node of degree 1. Then $K\backslash a = K' \subset K$, $|K'| \geqq 2$ and $|\mathscr{R} \cap \mathscr{P}(K')| = 2|K| - 3 - 1 = 2|K'| - 2 > 2|K'| - 3$. This contradicts property E.

Now assume all nodes to have degree $\geqq 4$. Then evidently $|\mathscr{R}| \geqq \frac{1}{2}$. $4|K|$ or $2|K| - 3 \geqq 2|K|$. So there is at least one node of order $\leqq 3$

Lemma 6.2: *Let* $\Gamma(K, \mathscr{R})$ *be an E-graph,* $L \subset K$, $M \subset K$, $|L \cap M| \geqq 2$, $|\mathscr{R} \cap \mathscr{P}(L)| = 2|L| - 3$, $|\mathscr{R} \cap \mathscr{P}(M)| = 2|M| - 3$. *Then* $|\mathscr{R} \cap \mathscr{P}(L \cup M)| = 2|L \cup M| - 3$.

Proof: $|\mathscr{R} \cap \mathscr{P}(L \cup M)| \geqq |\mathscr{R} \cap \mathscr{P}(L)| + |\mathscr{R} \cap \mathscr{P}(M)| - |\mathscr{R} \cap \mathscr{P}(L \cap M)| \geqq$

$$\geqq 2|L| - 3 + 2|M| - 3 - (2|L \cap M| - 3) = 2|L \cup M| - 3.$$

But by property E we also have $|\mathscr{R} \cap \mathscr{P}(L \cup M)| \leqq 2|L \cup M| - 3$. Therefore equality holds.

Theorem 6.3: *Let* $\Gamma(K, \mathscr{R})$ *be a graph and* $a \in K$ *a node of degree* 2. *Then* $\Gamma'(K\backslash a, \mathscr{R} \cap \mathscr{P}(K\backslash a))$ *is an E-graph if and only if* $\Gamma(K, \mathscr{R})$ *is an E-graph.*

Proof: If $\Gamma(K, \mathscr{R})$ is an E-graph we have

$$|\mathscr{R} \cap \mathscr{P}(K\backslash a)| = |\mathscr{R}| - 2 = 2|K| - 3 - 2 = 2|K\backslash a| - 3$$

and for every $L \subset K\backslash a$ with $|L| \geqq 2$

$$|\mathscr{R} \cap \mathscr{P}(K\backslash a) \cap \mathscr{P}(L)| = |\mathscr{R} \cap \mathscr{P}(L)| \leqq 2|L| - 3\,,$$

so Γ' is an E-graph.

If on the other hand $\Gamma'(K\backslash a, \mathscr{R} \cap \mathscr{P}(K\backslash a))$ is an E-graph, then

$$|\mathscr{R}| = |\mathscr{R} \cap \mathscr{P}(K\backslash a)| + 2 = 2|K\backslash a| - 3 + 2 = 2|K| - 3$$

and for $L \subset K$ with $|L| \geqq 2$

$$|\mathscr{R} \cap \mathscr{P}(L)| = |\mathscr{R} \cap \mathscr{P}(K\backslash a) \cap \mathscr{P}(L)| \leqq 2|L| - 3$$

in case $a \notin L$ and

$$|\mathscr{R} \cap \mathscr{P}(L)| = |\mathscr{R} \cap \mathscr{P}(K\backslash a) \cap \mathscr{P}(L)| + 2 \leqq 2|L\backslash a| - 3 + 2 = 2|L| - 3$$

in case $a \in L$. So Γ is shown to be an E-graph.

Theorem 6.4: *Let* $\Gamma(K, \mathscr{R})$ *be a graph and* $a \in K$ *a node of degree* 3. *Then the two following assertions are equivalent*:

A. $\Gamma(K, \mathscr{R})$ *is an E-graph.*

B. *To at least one of the three couples of neighbours of a—say* (b_1, b_2)*— there does not exist a* $L \in K\backslash a$ *with* $b_1 \in L$, $b_2 \in L$ *and* $|\mathscr{R} \cap \mathscr{P}(L)| = 2|L| - 3$ *and* $\Gamma'(K\backslash a, \mathscr{R}')$ *is an E-graph, where* $\mathscr{R}' = (\mathscr{R} \cap \mathscr{P}(K\backslash a)) \cup \{(b_1, b_2)\}$.

Proof:

I. A implies B. Indeed suppose there exist $L_i \subset K\backslash a$, $i = 1, 2, 3$ such that $b_2 \in L_1$, $b_3 \in L_1$, $b_3 \in L_2$, $b_1 \in L_2$, $b_1 \in L_3$, $b_2 \in L_3$ and $|\mathscr{R} \cap \mathscr{P}(L_i)| = 2|L_i| - 3$, $i = 1, 2, 3$. There are two cases:

1) Among 1, 2, 3 there are two indices—say 1 and 2—such that $|L_1 \cap L_2| \geqq 2$. Then by lemma 6.2 we have $|\mathscr{R} \cap \mathscr{P}(L_1 \cup L_2)| = 2|L_1 \cup L_2| - 3$. Moreover $|(L_1 \cup L_2) \cap L_3| \geqq 2$ (both contain b_1 and b_2). Applying lemma 6.2 once again we find $|\mathscr{R} \cap \mathscr{P}(L_1 \cup L_2 \cup L_3)| = 2|L_1 \cup L_2 \cup L_3| - 3$.

2) $L_1 \cap L_2|=|L_2 \cap L_3|=|L_3 \cap L_1|=1$. Then $|L_1|+|L_2|+|L_3|=|L_1 \cup L_2 \cup L_3|+3$, so $|\mathscr{R} \cap \mathscr{P}(L_1 \cup L_2 \cup L_3)| \geqq \Sigma_i |\mathscr{R} \cap \mathscr{P}(L_i)| = \Sigma_i (2|L_i|-3) = 2|L_1 \cup L_2 \cup L_3|+6-9 = 2|L_1 \cup L_2 \cup L_3|-3$.

In both cases we find

$$|\mathscr{R} \cap \mathscr{P}(L_1 \cup L_2 \cup L_3 \cup \{a\})| = 2|L_1 \cup L_2 \cup L_3|-3+3 > \\ > 2|L_1 \cup L_2 \cup L_3 \cup \{a\}|-3 .$$

This contradicts A and therefore the first part of B is proved. As to the second part of B we have $|\mathscr{R}'|=|\mathscr{R}|-3+1=2|K|-3-2=2|K\backslash a|-3$. Now let $M \subset K\backslash a$ and $|M| \geqq 2$. If $(b_1, b_2) \notin \mathscr{P}(M)$ then $|\mathscr{R}' \cap \mathscr{P}(M)|=|\mathscr{R} \cap \mathscr{P}(M)| \leqq 2|M|-3$ and if $(b_1, b_2) \in \mathscr{P}(M)$ then $|\mathscr{R} \cap \mathscr{P}(M)| < < 2|M|-3$ by the first part of B, so $|\mathscr{R}' \cap \mathscr{P}(M)|=|\mathscr{R} \cap \mathscr{P}(M)|+1 \leqq 2|M|-3$. Therefore Γ' is an E-graph.

II. B implies A. For $|\mathscr{R}|=|\mathscr{R} \cap \mathscr{P}(K\backslash a)|+3=|(\mathscr{R} \cap \mathscr{P}(K\backslash a)) \cup \{(b_1, b_2)\}|+2=2|K\backslash a|+ -3+2=2|K|-3$. And if $M \subset K$ with $|M| \geqq 2$ we prove $|\mathscr{R} \cap \mathscr{P}(M)| \leqq 2|M|-3$ in the three cases which present themselves.

1) If $a \notin M$ we have $|\mathscr{R} \cap \mathscr{P}(M)|=|\mathscr{R} \cap \mathscr{P}(K\backslash a) \cap \mathscr{P}(M)| \leqq |\{(\mathscr{R} \cap \mathscr{P}(K\backslash a)) \cup \{(b_1, b_2)\}\} \cap \mathscr{P}(M)| \leqq 2|M|-3$.
2) If $a \in M$ and either $b_1 \notin M$ or $b_2 \notin M$ then $|\mathscr{R} \cap \mathscr{P}(M)| \leqq |\mathscr{R} \cap \mathscr{P}(M\backslash a)|+2 \leqq 2|M\backslash a|+ -3+2=2|M|-3$.
3) If $a \in M$, $b_1 \in M$ and $b_2 \in M$ we have $|\mathscr{R} \cap \mathscr{P}(M)| \leqq |\mathscr{R} \cap \mathscr{P}(M\backslash a)|+3 < 2|M\backslash a|-3+3$ so $|\mathscr{R} \cap \mathscr{P}(M)| \leqq 2|M\backslash a|-1=2|M|-3$.

We can prove now the theorem announced at the end of Section 5:

Theorem 6.5: *Every E-graph has rigid realizations.*

Proof:

I. The (unique)-E-graph with $|K|=3$ evidently has rigid realizations.

II. Assume all E-graphs with k nodes to have rigid realizations and let $\Gamma(K, \mathscr{R})$ be an arbitrary E-graph with $|K|=k+1$. By proposition 6.1 there is either a node of degree 2 or a node of degree 3.

If a is a node of degree 2 then $\Gamma'(K\backslash a, \mathscr{R} \cap \mathscr{P}(K\backslash a))$ is an E-graph with $|K\backslash a|=k$ by theorem 6.3. By assumption Γ' has a rigid realization and by proposition 5.3 (Γ', χ') is extendable to a rigid realization (Γ, χ) of Γ.

If a is a node of degree 3 then theorem 6.4 yields an E-graph $\Gamma'(K', \mathscr{R}')$ with $|K'|=k$. By assumption Γ' has a rigid realization and by proposition 5.4 (Γ', χ') is extendable to a rigid realization (Γ, χ) of Γ.

By induction it follows that every E-graph has a rigid realization.